

# GROUP ACTIONS AND NON-KÄHLER COMPLEX MANIFOLDS

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**ABSTRACT.** New constructions of non-Kähler complex manifolds are presented. Let  $H$  be a complex linear algebraic group and let  $K$  be a maximal compact Lie subgroup of  $H$ . Let  $\mathcal{E}$  be a smooth principal  $K$ -bundle  $E_K \rightarrow M$  over a complex manifold  $M$ . If  $\mathcal{E}$  can be obtained by a smooth reduction of structure group from a holomorphic principal  $H$ -bundle over  $M$ , then  $E_K$  (respectively,  $E_K \times S^1$ ) admits an integrable complex structure if  $K$  has even dimension (respectively, odd dimension). As a consequence, the total space of the unitary frame bundle associated to any holomorphic vector bundle of even rank admits a complex analytic structure which is not Kähler. New complex manifolds are also derived from proper holomorphic actions of complex linear algebraic groups on complex manifolds. In particular, non-Kähler complex manifolds associated to effective complex analytic orbifolds are constructed.

## 1. INTRODUCTION

Let  $K$  be a compact connected Lie group, and let  $G$  be the universal complexification of  $K$ . Then  $G$  is a reductive complex Lie group and  $K$  is a maximal compact subgroup of  $G$  (cf. Section 5.3, [1]). In fact,  $G$  can be regarded as a reductive complex linear algebraic group.

We say that a smooth principal  $K$ -bundle  $E_K \rightarrow M$  over a complex manifold  $M$  admits a *complexification* if the associated principal  $G$ -bundle  $E_K \times_K G \rightarrow M$  admits a holomorphic principal  $G$ -bundle structure. Note that if  $E_K \rightarrow M$  admits a complexification, then the transversely holomorphic foliation induced by the  $K$ -action on  $E_K$  admits a complexification in the sense of Haefliger-Sundaraman (cf. Remark 4.3, [8]).

More generally, let  $H$  be a complex linear algebraic group and  $K$  be a maximal compact subgroup of  $H$ . Then  $K$  is a smooth deformation retract of  $H$  (cf. Theorem 3.4, Chapter 4, [14]). Therefore, any holomorphic principal  $H$ -bundle  $E_H \rightarrow M$  over a complex manifold  $M$ , when considered as a smooth bundle, admits a smooth reduction of the structure group from  $H$  to  $K$ . The smooth principal  $K$ -bundle  $E_K \rightarrow M$ , corresponding to this reduction, admits a complexification (see Proposition 3.2 below).

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Let  $E_K \rightarrow M$  be a smooth principal  $K$ -bundle which admits a complexification. Suppose  $K$  is even dimensional. Then we show that  $E_K \rightarrow M$  has a holomorphic fiber bundle structure. In addition, if  $K$  is a compact torus then the bundle  $E_K \rightarrow M$  has a holomorphic principal  $K$ -bundle structure. Similarly, if  $K$  is odd dimensional, the bundle  $E_K \times S^1 \rightarrow M$  admits a holomorphic fiber bundle structure with fiber  $K \times S^1$ . In particular, the space  $E_K$  admits a normal almost contact structure (nacs). Moreover, in most cases, the space  $E_K$  or  $E_K \times S^1$  is a non-Kähler complex manifold. In this way we obtain new examples of non-Kähler complex manifolds; see Corollary 3.5, Examples 3.6, 3.10, and Remark 3.12 below.

The above construction of non-Kähler complex manifolds generalizes some well-known constructions in the literature. For example, Hopf [11] and Calabi-Eckmann [6] constructed compact non-Kähler complex manifolds by obtaining complex analytic structures on the product of two odd dimensional spheres,  $S^{2m-1} \times S^{2n-1}$ . Such a product can be viewed as the total space of a smooth principal  $S^1 \times S^1$ -bundle which admits a complexification. To see this, consider the holomorphic principal  $\mathbb{C}^* \times \mathbb{C}^*$ -bundle,

$$(\mathbb{C}^m \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\}) \rightarrow \mathbb{CP}^{m-1} \times \mathbb{CP}^{n-1}. \quad (1.1)$$

Here, the projection map is the component-wise Hopf map. The above principal bundle, when considered as a smooth bundle, admits a smooth reduction of structure group from  $\mathbb{C}^* \times \mathbb{C}^*$  to  $S^1 \times S^1$  to yield a smooth principal  $S^1 \times S^1$ -bundle

$$S^{2m-1} \times S^{2n-1} \rightarrow \mathbb{CP}^{m-1} \times \mathbb{CP}^{n-1}. \quad (1.2)$$

Hence, the smooth principal  $S^1 \times S^1$ -bundle (1.2) has a complexification, and it admits the structure of a holomorphic principal bundle with an elliptic curve as the structure group. The complex structures thus obtained on  $S^{2m-1} \times S^{2n-1}$  are those of Hopf and Calabi-Eckmann manifolds.

Recently, there have been many interesting generalizations of Hopf and Calabi-Eckmann manifolds, giving new classes of non-Kähler compact complex manifolds. Bosio [4] obtained such a class of manifolds which are now known as LVMB manifolds. These manifolds were obtained by generalizing earlier constructions (of LVM manifolds) by López de Madrano-Verjovsky [17] and Meersseman [18]. Bosio and Meersseman observed that the underlying smooth manifolds of a large class of these are polytopal moment angle manifolds [5]. Complex analytic structures were then constructed on a more general class of moment angle manifolds by Panov and Ustinovskiy [22]. In [24], inspired by Loeb-Nicolau's construction [15], Sankaran and the second-named author obtained a family of complex structures on  $S(\mathcal{L}_1) \times S(\mathcal{L}_2)$ , where  $S(\mathcal{L}_i) \rightarrow X_i$  is the smooth principal  $S^1$ -bundle associated to a holomorphic principal  $\mathbb{C}^*$ -bundle  $\mathcal{L}_i \rightarrow X_i$  over a complex manifold  $X_i$ ,  $i = 1, 2$ .

By the description of Cupit-Foutou and Zaffran [28], an LVMB manifold satisfying a certain rationality condition (K) admits a Seifert principal fibration over an orbifold toric variety. If the base of the Seifert principal fibration is a nonsingular toric variety

then the corresponding LVMB manifold can be viewed as the total space of a smooth principal bundle which admits a complexification. Hence, we recover these LVMB manifolds as a special case of the construction described above.

It is possible to look at all LVMB manifolds that satisfy condition (K) as part of a more general phenomenon. It is shown in [28] that any such LVMB manifold can be identified to an orbit space  $X/L$ , where  $X$  is a complex manifold that admits a proper holomorphic action of  $(\mathbb{C}^*)^{2m}$  with finite stabilizers, and  $L$  is a torsion-free closed co-compact subgroup of  $(\mathbb{C}^*)^{2m}$ . In Section 4, we consider the more general situation of any proper holomorphic action of a complex linear algebraic group  $H$  on a complex manifold  $X$ . If the rank of  $H$  is greater than one, then  $H$  has a nontrivial closed torsion-free complex Lie subgroup  $L$  which acts freely on  $X$ . This yields new, possibly non-Kähler, complex manifolds  $X/L$ . In particular, given an even dimensional effective complex analytic orbifold  $V^{2n}$ , we can take  $X$  to be the manifold of frames of the holomorphic tangent (or cotangent) bundle of  $V^{2n}$ , and  $G$  to be  $GL(2n, \mathbb{C})$ . Then, there exists  $L$  such that  $G/L$  is diffeomorphic to  $U(2n)$ , and  $X/L$  is a non-Kähler complex manifold. Analogous results are obtained in the case of odd dimensional Calabi-Yau orbifolds.

## 2. COMPLEX STRUCTURES ON AN EVEN DIMENSIONAL COMPACT LIE GROUP

Samelson [12] and Wang [27] proved the existence of a family of left invariant complex analytic structures on an even dimensional compact Lie group. A classification of such structures was given in [23]. Recently, the topic was revisited in [16] where complex analytic structures that are not invariant were also constructed. We briefly review the left invariant complex structures following [23] and [16].

Let  $K$  be an even dimensional compact connected Lie group of rank  $2r$ . Let  $\mathfrak{k}$  be the Lie algebra of  $K$ . Let  $G$  be the universal complexification of  $K$  and  $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$  be its Lie algebra. By Proposition 2.2, [23], a left invariant complex structure on  $K$  is determined by a complex Lie subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  such

$$\mathfrak{l} \cap \mathfrak{k} = 0 \text{ and } \mathfrak{g} = \mathfrak{l} \oplus \bar{\mathfrak{l}}. \quad (2.1)$$

Pittie [23] called a complex Lie subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  satisfying condition 2.1 a Samelson subalgebra. Our main interest is in the complex Lie subgroup  $L$  of  $G$  corresponding to a Samelson subalgebra  $\mathfrak{l}$ . We assume that the Lie subalgebra  $\mathfrak{l}$  defines vectors of type  $(0, 1)$  in  $T_e(K) \otimes \mathbb{C} = \mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$ . This is necessary for the conformity of the complex structures on  $K$  and  $G/L$  in Proposition 2.1 below.

Fix a maximal torus  $T \cong (S^1)^{2r}$  of  $K$  and let  $B$  be the Borel subgroup of  $G$  containing  $T$ . Let  $\mathfrak{t}$  and  $\mathfrak{b}$  be the Lie algebras of  $T$  and  $B$  respectively. Let  $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$  be the Cartan subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{t}$ . Let  $\mathfrak{a}$  be a subalgebra of  $\mathfrak{h}$  such that  $\mathfrak{h} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$ . Then

$$\mathfrak{l} := \mathfrak{a} \oplus [\mathfrak{b}, \mathfrak{b}] \quad (2.2)$$

is a Samelson subalgebra, and every Samelson subalgebra is of this form for appropriate choices of  $\mathfrak{t}$ ,  $\mathfrak{b}$  and  $\mathfrak{a}$  (cf. Corollary 2.5.1, [23]).

For fixed  $\mathfrak{t}$  and  $\mathfrak{b}$ , the possible choices of  $\mathfrak{a}$  are described below. Fix a real basis for  $\mathfrak{t}$  and identify  $\mathfrak{h}$  with  $\mathbb{C}^{2r}$  so that  $\mathfrak{t}$  is included in  $\mathfrak{h}$  as  $i\mathbb{R}^{2r} \subset \mathbb{C}^{2r}$ . Now consider a  $\mathbb{C}$ -linear map  $\Lambda : \mathbb{C}^r \rightarrow \mathfrak{h}$  given by

$$z \mapsto A \cdot z$$

where  $A = (a_i^j)$  is a  $(2r \times r)$ -complex matrix. We denote the image of this linear map by  $\mathfrak{a}$ . Let

$$A_\Lambda := \begin{pmatrix} \operatorname{Re} a_1^1 & -\operatorname{Im} a_1^1 & \cdots & \operatorname{Re} a_1^r & -\operatorname{Im} a_1^r \\ \vdots & \vdots & & \vdots & \vdots \\ \operatorname{Re} a_{2r}^1 & -\operatorname{Im} a_{2r}^1 & \cdots & \operatorname{Re} a_{2r}^r & -\operatorname{Im} a_{2r}^r \end{pmatrix}$$

be the  $(2r \times 2r)$ -real matrix where  $\operatorname{Re} a_i^j$  and  $\operatorname{Im} a_i^j$  denote the real and imaginary parts of  $a_i^j$  respectively. Note that  $\mathfrak{a} \cap \mathfrak{t} = 0$  if and only if  $A_\Lambda$  is non-degenerate (cf. Lemma 3.1 [16]). In this case  $\mathfrak{h} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$ . Moreover, any such decomposition of  $\mathfrak{h}$  corresponds to a  $\mathbb{C}$ -linear map  $\Lambda : \mathbb{C}^r \rightarrow \mathfrak{h}$  with  $\det A_\Lambda \neq 0$ .

Let  $H \cong (\mathbb{C}^*)^{2r}$  be the Cartan subgroup of  $G$  associated to  $\mathfrak{h}$ . Let  $\exp(\mathfrak{a})$  be the connected complex Lie subgroup of  $H$  with Lie algebra  $\mathfrak{a}$ . Then

$$\dim(\exp(\mathfrak{a}) \cap (S^1)^{2r}) = 0$$

as  $\mathfrak{a} \cap \mathfrak{t} = 0$ . In this case, it follows by Lemma 2.9 [16] that  $\exp(\mathfrak{a})$  is isomorphic to  $\mathbb{C}^r$ . By compactness of  $(S^1)^{2r}$ ,  $\exp(\mathfrak{a}) \cap (S^1)^{2r}$  is finite. Since  $\exp(\mathfrak{a})$  is torsion-free, we have that  $\exp(\mathfrak{a}) \cap (S^1)^{2r}$  is trivial. Furthermore,  $\exp(\mathfrak{a})$  is a closed subgroup of  $H$  (cf. Lemma 3.1, [16]).

Note that the subalgebra  $[\mathfrak{b}, \mathfrak{b}]$  is an ideal in  $\mathfrak{b}$ . The connected complex Lie subgroup  $U$  of  $G$  corresponding to  $[\mathfrak{b}, \mathfrak{b}]$  is the unipotent radical of  $B$  and we have the semidirect product decomposition,  $B = H \cdot U$ . Hence the connected complex Lie subgroup  $L$  of  $G$  associated to the Samelson subalgebra  $\mathfrak{l} = \mathfrak{a} \oplus [\mathfrak{b}, \mathfrak{b}]$  is of the form

$$L = \exp(\mathfrak{a}) \cdot U.$$

As  $U$  is torsion-free and contractible (cf. Proposition 8.2.1 [25]), it follows that  $L$  is a contractible, closed and torsion-free subgroup of  $G$ .

Let  $\tilde{G}$  be the simply-connected complex Lie group with Lie algebra  $\mathfrak{g}$  and let  $\tilde{L}$  be the connected closed Lie subgroup of  $\tilde{G}$  associated to Lie algebra  $\mathfrak{l}$ . Then  $\tilde{L}$  is a covering group of  $L$ . However  $L$  is contractible. So  $\tilde{L}$  is isomorphic to  $L$ . The proof of the following proposition is now evident from Proposition 2.3 [23] and the discussion below it.

**Proposition 2.1.** *Let  $K$  be an even dimensional connected compact Lie group and let  $G$  be the universal complexification of  $K$ . Let  $K$  be endowed with the left invariant complex structure define by a Samelson subalgebra  $\mathfrak{l}$ . Let  $L$  be the connected complex*

Lie subgroup of  $G$  associated to  $\mathfrak{l}$ . Then  $L$  is a torsion-free closed subgroup of  $G$  such that  $K \cap L = 1_G$  and  $KL = G$ . Moreover, the natural map  $K \rightarrow G/L$  is a biholomorphism.  $\square$

The above proposition shows that there is a holomorphic principal  $L$ -bundle,

$$L \hookrightarrow G \rightarrow K.$$

### 3. PRINCIPAL BUNDLES

Let  $K$  be a compact connected Lie group of even dimension. Let  $G$  be a complex Lie group that contains  $K$  as a real Lie subgroup. Let  $L$  be a closed complex Lie subgroup of  $G$  such that  $K \cap L = 1_G$  and  $KL = G$ . Then the natural map  $K \rightarrow G/L$  is a diffeomorphism and this induces a complex structure on  $K$ .

In these notes, by  $L$ -foliation we mean a foliation with leaf  $L$ . Consider the left  $G$ -invariant  $L$ -foliation on  $G$ , given by the right translation action of  $L$  on  $G$ . Note that this foliation is transverse to  $K$ .

**Proposition 3.1.** *Let  $K, G$  and  $L$  be as defined above. Let  $E_G \rightarrow M$  be a holomorphic principal  $G$ -bundle over a complex manifold  $M$ . Assume that  $E_G \rightarrow M$  admits a smooth reduction of structure group to give a smooth principal  $K$ -bundle,  $E_K \rightarrow M$ . Then the total space  $E_K$  admits a complex structure such that  $E_K \rightarrow M$  is a holomorphic fiber bundle with fiber  $K$ . Further, if  $G$  is abelian then  $E_K \rightarrow M$  is a holomorphic principal  $K$ -bundle.*

*Proof.* Since the foliation on  $G$  corresponding to the right action of  $L$  is left  $G$ -invariant, it induces an  $L$ -foliation on  $E_G$ . This foliation is complex analytic as  $E_G$  is a holomorphic bundle. Similarly, there is an induced smooth  $L$ -foliation on  $E_K \times_K G$ .

Let  $i : E_K \hookrightarrow E_K \times_K G$  be the inclusion map defined by  $i(e) = [e, 1_G]$ . Since  $E_K \rightarrow M$  is obtained by a smooth reduction of structure group from  $E_G \rightarrow M$ , there exists a smooth isomorphism of principal  $G$ -bundles,  $\rho : E_K \times_K G \rightarrow E_G$ . Then  $\rho \circ i$  is a smooth embedding of  $E_K$  into  $E_G$ .

The image of  $i$  is transversal to the  $L$ -foliation in  $E_K \times_K G$ . Since the map  $\rho$  is  $L$ -equivariant, it respects the fiberwise  $L$ -foliations on  $E_K \times_K G$  and  $E_G$ . Being a (fiberwise) diffeomorphism,  $\rho$  preserves transversality. Therefore, the  $L$ -foliation in every fiber of  $E_G$  is transversal to the image of corresponding fiber of  $E_K$  under  $\rho \circ i$ . It follows that the  $L$ -foliation on  $E_G$  is transversal to the image of  $E_K$  under  $\rho \circ i$ . Moreover,  $L$  is closed and  $K \cap L = 1_G$ . Therefore, the map  $E_K \rightarrow E_G/L$  induced by  $\rho \circ i$  is a diffeomorphism. Hence, the holomorphic fiber bundle structure on  $E_G/L \rightarrow M$  can be pulled back to give a holomorphic fiber bundle structure on  $E_K \rightarrow M$ .

Finally, if  $G$  is abelian, then  $G/L$  inherits the structure of a complex Lie group. This induces a complex Lie group structure on  $K$ . With respect to this structure  $E_K \rightarrow M$  is a holomorphic principal  $K$ -bundle.  $\square$

Let  $H$  be a complex linear algebraic group. Let  $K$  be a maximal compact subgroup of  $H$ . As  $K$  is a smooth deformation retract of  $H$  (cf. Theorem 3.4, Chapter 4, [14]), a holomorphic principal  $H$ -bundle,  $E_H \rightarrow M$ , over a complex manifold  $M$  admits a smooth reduction of the structure group from  $H$  to  $K$ .

**Proposition 3.2.** *Let  $H$  be a complex linear algebraic group and let  $K$  be a maximal compact subgroup of  $H$ . Let  $E_H \rightarrow M$  be a holomorphic principal  $H$ -bundle. Let  $E_K \rightarrow M$  be a smooth principal  $K$ -bundle corresponding to the smooth reduction of the structure group from  $H$  to  $K$ . Then  $E_K \rightarrow M$  admits a complexification.*

*Proof.* Let  $G$  be a reductive Levi subgroup of  $H$  containing  $K$ . Then  $G$  is a maximal reductive subgroup of  $H$  and it is the universal complexification of the compact Lie group  $K$ . Further, the algebraic group  $H$  is isomorphic to the semi-direct product  $G \cdot U$ , where  $U$  is the unipotent radical of  $H$  (cf. Chapter 6, [21]).

Consider the right action of  $U$  on  $E_H$ . The induced map  $E_H/U \rightarrow M$  defines a holomorphic principal  $G$ -bundle. Observe that the smooth principal  $G$ -bundles  $E_K \times_K G \rightarrow M$  and  $E_H/U \rightarrow M$  are smoothly isomorphic.  $\square$

We now state our main theorem.

**Theorem 3.3.** *Let  $K$  be an even dimensional compact connected Lie group endowed with a left invariant complex structure. Let  $E_K \rightarrow M$  be a smooth principal  $K$ -bundle over a complex manifold  $M$ , which admits a complexification. Then the total space  $E_K$  admits a complex structure such that  $E_K \rightarrow M$  is a holomorphic fiber bundle with fiber  $K$ . Further, if  $K$  is abelian then  $E_K \rightarrow M$  has the structure of a holomorphic principal  $K$ -bundle.*

*Proof.* The result follows by applying Propositions 3.1 and 2.1. Note that if  $K$  is abelian, then its universal complexification is also abelian.  $\square$

In most of the cases the complex manifold  $E_K$  is non-Kähler. First assume that  $K$  is a non-abelian compact Lie group. Then, it is shown in Example 2.10, [19] that  $K$  is non-Kähler with respect to any complex structure. We reproduce the argument given there: Any compact connected Lie group  $K$  can be written as

$$K = K_1 \times (S^1)^r / \Gamma$$

where  $K_1$  is a semi-simple compact Lie group,  $\Gamma$  is a finite subgroup of the center  $Z(K_1 \times (S^1)^r)$  of  $K_1 \times (S^1)^r$  and  $r \geq 0$ . Since the first and the second Betti numbers of  $K_1$  are zero (cf. [13, p. 29]), we have  $H^2(K_1 \times (S^1)^r; \mathbb{C}) = H^2((S^1)^r; \mathbb{C})$ . This implies that there does not exist a class  $[\omega] \in H^2(K_1 \times (S^1)^r; \mathbb{C})$  such that

$$[\omega]^{\dim(K_1 \times (S^1)^r)} \neq 0.$$

Hence  $K_1 \times (S^1)^r$  is non-Kähler. From here one easily concludes that any non-abelian even dimensional compact Lie group  $K$  cannot be Kähler.

**Theorem 3.4.** *Assume that  $K$  is a non-abelian compact connected Lie group of even dimension. Then the complex manifold  $E_K$  of Theorem 3.3 is non-Kähler.*

*Proof.* Any fiber of the holomorphic fiber bundle  $E_K \rightarrow M$  is a complex submanifold of the total space  $E_K$ . Each fiber is diffeomorphic to  $K$ , and hence non-Kähler. The theorem follows since a closed complex submanifold of a Kähler manifold is Kähler.  $\square$

The following corollary and example yield new classes of non-Kähler complex manifolds.

**Corollary 3.5.** *Let  $E \rightarrow M$  be a holomorphic vector bundle of rank  $2n$  over a complex manifold  $M$ . Let  $U(E) \rightarrow M$  denote the unitary frame bundle associated to the vector bundle  $E \rightarrow M$ . Then  $U(E)$  is a non-Kähler complex manifold. Moreover,  $U(E) \rightarrow M$  has the structure of a holomorphic fiber bundle.*

*Proof.* The frame bundle associated to the vector bundle  $E \rightarrow M$  is a holomorphic principal  $GL(2n, \mathbb{C})$ -bundle. Then the unitary frame bundle  $U(E) \rightarrow M$  is a smooth principal  $U(2n)$ -bundle corresponding to the smooth reduction of structure group from  $GL(2n, \mathbb{C})$  to  $U(2n)$ . As  $GL(2n, \mathbb{C})$  is the universal complexification of  $U(2n)$ , by Theorem 3.3, the bundle  $U(E) \rightarrow M$  has the structure of a holomorphic fiber bundle. The total space  $U(E)$  is not Kähler by Theorem 3.4.  $\square$

**Example 3.6.** Let  $M$  be a complex  $n$ -dimensional Calabi-Yau manifold, where  $n$  is odd. Then  $M$  is a Kähler manifold that admits a non-vanishing holomorphic  $n$ -form  $\Omega$ . This implies that the holomorphic cotangent bundle  $T^{*(1,0)}M$  of  $M$  admits a holomorphic reduction of structure group to  $SL(n, \mathbb{C})$ . Since  $SU(n)$  is a maximal compact subgroup of  $SL(n, \mathbb{C})$ ,  $T^{*(1,0)}M$  admits a smooth reduction of structure group to  $SU(n)$ . Note that  $SU(n)$  is even dimensional when  $n$  is odd. Let  $G = SL(n, \mathbb{C})$  and  $K = SU(n)$ . Let  $E_G \rightarrow M$  be the bundle of precisely those frames of  $T^{*(1,0)}M$  the wedge product of whose components equals  $\Omega$ . Then  $E_G \rightarrow M$  is a holomorphic principal  $G$ -bundle. Let  $E_K \rightarrow M$  be the associated principal  $K$ -bundle corresponding to the smooth reduction of structure group from  $G$  to  $K$ . Then by Theorems 3.3 and 3.4, the total space  $E_K$  is a non-Kähler complex manifold. Note that  $E_K$  is smoothly isomorphic to the bundle of special unitary frames of  $T^{*(1,0)}M$ .

By duality, the form  $\Omega$  induces a holomorphic trivialization of the line bundle  $\wedge^n T^{(1,0)}M$ . Therefore  $T^{(1,0)}M$  admits holomorphic reduction of structure group to  $SL(n, \mathbb{C})$ . So, by a similar argument as above, the space of special unitary frames of  $T^{(1,0)}M$  admits the structure of a non-Kähler complex manifold.

When  $K$  is an even dimensional compact torus, we observe that the complex manifold  $E_K$  is again non-Kähler under certain conditions. In this case, the universal complexification  $G$  of  $K$  is a complex torus  $(\mathbb{C}^*)^{2r}$ . Then the rank  $2r$  vector bundle, corresponding to any holomorphic principal  $G$ -bundle  $E_G \rightarrow M$ , is the direct

sum of  $2r$  holomorphic line bundles,  $\mathcal{L}_1, \dots, \mathcal{L}_{2r}$ . The following result follows from Proposition 11.3 [9] and Corollary 11.4 [9].

**Theorem 3.7** (Höfer). *Assume that  $K$  is an even dimensional, compact torus and  $M$  is a simply-connected compact complex manifold. If the Chern classes  $c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_{2r}) \in H^2(M, \mathbb{R})$  are  $\mathbb{R}$ -linearly independent then the complex manifold  $E_K$  of Theorem 3.3 is not symplectic, and hence not Kähler.*  $\square$

In the case when  $M$  is Kähler, Blanchard [3] (cf. Section 1.7, [9]) gives the following necessary and sufficient condition.

**Theorem 3.8** (Blanchard). *Assume that  $K$  is an even dimensional, compact torus and  $M$  is a compact Kähler manifold. Then the complex manifold  $E_K$  of Theorem 3.3 is Kähler if and only if  $c_1(\mathcal{L}_i) \in H^2(M, \mathbb{R})$  is zero for each  $i$ .*  $\square$

If  $K$  is an elliptic curve, the following gives a sufficient condition without assuming the compact complex manifold  $M$  to be simply-connected or Kähler (cf. Corollary 1, [29]).

**Theorem 3.9** (Vuletescu). *Assume that  $K$  is an elliptic curve and at least one of the Chern classes  $c_1(\mathcal{L}_i) \in H^2(M, \mathbb{R})$  is non-zero. Then the complex manifold  $E_K$  of Theorem 3.3 is a non-Kähler complex manifold.*  $\square$

**Example 3.10.** Consider a nonsingular toric variety  $M$  of complex dimension  $n$ . Assume that the one dimensional cones in the fan  $\Sigma$  of  $M$  are generated by primitive integral vectors  $\rho_1, \dots, \rho_k$ . Let  $R$  be the matrix  $[\rho_1 \dots \rho_k]$ . Let  $\mathcal{E}_j$  be the line bundle over  $M$  corresponding to the  $j$ -th one dimensional cone. Then any algebraic line bundle over  $M$  is of the form  $\bigotimes \mathcal{E}_j^{a_j}$  where each  $a_j$  is an integer.

The first Chern class of such a line bundle is given by

$$c_1(\bigotimes \mathcal{E}_j^{a_j}) = \sum a_j c_1(\mathcal{E}_j).$$

It is zero if and only if the vector  $(a_1, \dots, a_k)$  belongs to the row space of the matrix  $R$ .

Consider an algebraic vector bundle  $E \rightarrow M$  which is the direct sum of an even number, namely  $2r$ , of line bundles. Let  $K = (S^1)^{2r}$ . Let  $E_K \rightarrow M$  denote the smooth principal  $K$ -bundle obtained by reduction of structure group from the principal holomorphic  $(\mathbb{C}^*)^{2r}$ -bundle associated to  $E$ . The total space  $E_K$  of this bundle admits a family of complex analytic structures by Theorem 3.3.

If  $M$  is projective, then Theorem 3.8 gives a sufficient condition for the total space  $E_K$  to be non-Kähler. The moment angle manifold corresponding to the fan  $\Sigma$  (cf. [22]) is an example of such an  $E_K$ .

Now, we discuss the case when the compact Lie group  $K$  is odd dimensional. We refer to [2] for the definition of a normal almost contact structure (nacs) on a smooth manifold. We have the following result.



**Corollary 3.11.** *Let  $K$  be an odd dimensional compact connected Lie group. Let  $K \times S^1$  be endowed with a left invariant complex structure. If a smooth principal  $K$ -bundle  $E_K \rightarrow M$  over a complex manifold  $M$  admits a complexification, then the bundle  $E_K \times S^1 \rightarrow M$  has the structure of a holomorphic fiber bundle with fibre  $K \times S^1$ . In particular, the space  $E_K$  admits a normal almost contact structure.*

*Proof.* Let  $E_G \rightarrow M$  denote the complexification of  $E_K \rightarrow M$ . Then  $E_G \times \mathbb{C}^* \rightarrow M$  is the complexification of the principal  $(K \times S^1)$ -bundle  $E_K \times S^1 \rightarrow M$ . Since  $K \times S^1$  is even dimensional, the proof of the first part of the corollary follows from Theorem 3.3. The second part of the corollary follows from the fact that the inclusion map  $E_K \rightarrow E_K \times \{0\} \subset E_K \times \mathbb{R}$  is an embedding of  $E_K$  as an orientable real hypersurface of the complex manifold  $E_K \times \mathbb{R}$  (cf. Example 4.5.2 and Section 6.1 of [2]).  $\square$

With regard to the above corollary, we note that a nacs endows a CR-structure of hypersurface type on  $E_K$  (cf. Theorem 6.6 [2]).

**Remark 3.12.** Let  $K$  and  $E_K$  be as in Corollary 3.11. If  $K$  is non-abelian, then  $E_K \times S^1$  is a non-Kähler complex manifold. On the other hand, if  $K$  is a compact torus, then Theorems 3.8 and 3.9 give some conditions for  $E_K \times S^1$  to be non-Kähler.

#### 4. PROPER ACTION OF A COMPLEX LINEAR ALGEBRAIC GROUP

Consider any proper holomorphic right action of a complex Lie group  $H$  on a complex manifold  $X$ . A stabilizer  $H_x, x \in X$ , of this action is a compact Lie subgroup of  $H$ . Suppose  $L$  is a torsion-free closed complex Lie subgroup of  $H$ . Then  $H_x \cap L$  is a compact torsion-free Lie group, and hence trivial. Therefore, the induced action of  $L$  on  $X$  is proper and free. Then the quotient  $X/L$  has the structure of a complex manifold such that the projection  $X \rightarrow X/L$  is a holomorphic map. If the  $H$ -action on  $X$  is also free, we get a holomorphic fiber bundle  $X/L \rightarrow X/H$  with fiber  $H/L$ . Further, if  $H$  is abelian then the holomorphic fiber bundle  $X/L \rightarrow X/H$  has the structure of a holomorphic principal  $(H/L)$ -bundle.

**Lemma 4.1.** *Let  $H$  be a complex linear algebraic group with  $\text{rank}(H) > 1$  and let  $K$  be a maximal compact subgroup of  $H$ . Then there exists a nontrivial torsion-free closed complex Lie subgroup  $L$  of  $H$ . Further, if  $H$  is of even rank, then we can choose  $L$  to be transverse to  $K$ .*

*Proof.* First assume that  $H$  is a reductive complex linear algebraic group. Then the lemma follows from Proposition 2.1 if the maximal compact subgroup  $K$  of  $H$  is even dimensional. Otherwise, note that  $K \times S^1$  is an even-dimensional maximal compact Lie subgroup of the reductive group  $H \times \mathbb{C}^*$ . Then by Proposition 2.1,  $H \times \mathbb{C}^*$  has a torsion-free closed complex Lie subgroup  $L'$  such that  $H \times \mathbb{C}^* = (K \times S^1)L'$ . This implies  $H \cap L'$  is a torsion-free closed complex Lie subgroup of  $H$ . Moreover, by dimension counting,  $L := H \cap L'$  is nontrivial.

Now assume that  $H$  is a complex linear algebraic group. As noted in the previous section,  $H$  is isomorphic to the semi-direct product  $G \cdot U$ , where  $U$  is the unipotent radical of  $H$  and  $G \cong H/U$  is a reductive Levi subgroup of  $H$  containing  $K$ . The complex reductive Lie group  $G$  contains  $K$  as a maximal compact Lie subgroup. If  $\pi : H \rightarrow H/U$  is the projection map, then the inverse image  $\pi^{-1}(L')$  of a torsion-free closed complex Lie subgroup  $L' \in H/U \cong G$  is a torsion-free closed complex Lie subgroup of  $H$ . This completes the proof.  $\square$

By the above lemma, any proper holomorphic action of a complex linear algebraic group  $H$  of rank  $> 1$  on a complex manifold  $X$  yields a complex manifold  $X/L$ . Any LVMB manifold satisfying condition (K) (cf. [28]), and all the examples in section 3 are of the form  $X/L$  as above. Further, in all these examples the corresponding group  $H$  acts with at most finite stabilizers so that the quotient  $X/H$  has the structure of a complex analytic orbifold.

Now consider any  $n$ -dimensional, effective, complex analytic orbifold  $V$ . We will construct some complex manifolds associated to  $V$ .

Note that  $V$  is the quotient of a complex manifold  $X$  by a proper, holomorphic action of  $GL(n, \mathbb{C})$ . This follows from the so-called frame bundle construction: Let  $\{V_i : i \in I\}$  be an open cover of  $V$ . Let  $\{(U_i, \Gamma_i, q_i)\}$  be an orbifold atlas for  $V$  corresponding to this cover. Here we may assume that

- $U_i \subset \mathbb{C}^n$ ,
- $\Gamma_i$  is a finite subgroup of complex analytic isomorphisms of  $U_i$  (In fact,  $\Gamma_i$  can be assumed to be a finite subgroup of  $GL(n, \mathbb{C})$  by [7]),
- $q_i : U_i \rightarrow V_i$  is a continuous map that induces a homeomorphism  $\Gamma_i \backslash U_i \rightarrow V_i$ .

Let  $\tilde{X}_i$  be the space of frames of the holomorphic tangent bundle of  $U_i$ . Define  $X_i = \Gamma_i \backslash \tilde{X}_i$ . Then the  $X_i$  glue together naturally to give a complex manifold  $X$ . For details, we refer to [30] where an analogous construction is described in the presence of a hermitian metric. There is a natural right action of  $GL(n, \mathbb{C})$  on  $X$ , which is holomorphic and proper (but not free unless  $V$  is a manifold). The quotient  $X/GL(n, \mathbb{C})$  is isomorphic to  $V$  as a complex orbifold.

Assume  $n$  is even. Then  $U(n)$  has even dimension. Since  $U(n)$  is a maximal compact subgroup of  $GL(n, \mathbb{C})$ , by Proposition 2.1 there exists a closed and torsion-free complex Lie subgroup  $L$  of  $GL(n, \mathbb{C})$  such that  $GL(n, \mathbb{C})/L$  is diffeomorphic to  $U(n)$ . In this case, the complex manifold  $X/L$  is diffeomorphic to the space of unitary frames,  $\tilde{V}$ , of [30]. The fiber of the map  $p : X/L \rightarrow V$  at a regular point of  $V$  is biholomorphic to  $GL(n, \mathbb{C})/L$ . The fiber is indeed a complex submanifold of  $X/L$ . Since  $U(n)$  is not Kähler,  $X/L$  can not be Kähler.

Now, assume  $n$  is odd. Then  $SU(n)$  is even dimensional. Suppose the complex orbifold  $V$  is Calabi-Yau. This means there is a holomorphic volume form  $\Omega$  on  $V$ . So each local chart  $U_i$  has a holomorphic volume form  $\Omega_i$  which is invariant under  $\Gamma_i$ , and these forms are compatible under gluing of orbifold charts. In particular each  $\Gamma_i$

is a subgroup of  $SL(n, \mathbb{C})$ . We define  $\tilde{X}_i^*$  to be the space of those frames of  $T^{*(1,0)}U_i$  the wedge product of whose components equals  $\Omega_i$ . Let  $X_i^* = \Gamma_i \backslash \tilde{X}_i^*$ . Then the  $X_i^*$  glue together to produce a complex manifold  $X^*$  which admits a proper holomorphic action of  $SL(n, \mathbb{C})$ . By a similar argument as above, there exists a complex subgroup  $L$  of  $SL(n, \mathbb{C})$  such that  $SL(n, \mathbb{C})/L \cong SU(n)$  and  $X^*/L$  is a complex non-Kähler manifold. Note that, by duality,  $\Omega$  induces a holomorphic trivialization of  $\wedge^n T^{(1,0)}V$ . Here  $T^{(1,0)}V$  denotes the holomorphic tangent bundle of  $V$  whose sections over  $V_i$  are the  $\Gamma_i$ -invariant sections of  $T^{(1,0)}U_i$ . Therefore, there exists an invariant, nowhere vanishing, holomorphic poly-vector field  $\eta$  of degree  $n$  on  $V$ . Therefore, an analogous construction can be made with tangent frames whose components have wedge product  $\eta$ .

**Remark 4.2.** If we consider  $V$  to be an orbifold toric variety, then the manifolds  $X/L$  and  $X^*/L$  are different from the LVMB manifolds associated to  $V$ .

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## REFERENCES

- [1] D. N. Akhiezer: Lie group actions in complex analysis. Aspects of Mathematics, E27. Friedr. Vieweg & Sohn, Braunschweig, 1995. viii+201 pp.
- [2] D. E. Blair: Riemannian geometry of contact and symplectic manifolds. Progress in Mathematics, 203. Birkhuser Boston, Inc., Boston, MA, 2010.
- [3] A. Blanchard: Variétés kählériennes et espaces fibrés. C. R. Acad. Sci. Paris **234** (1952), 284-286.
- [4] F. Bosio: Variétés complexes compactes: une généralisation de la construction de Meersseman et López de Medrano-Verjovsky. Ann. Inst. Fourier (Grenoble) **51** (2001), no. 5, 1259-1297.
- [5] F. Bosio and L. Meersseman: Real quadrics in  $\mathbb{C}n$ , complex manifolds and convex polytopes. Acta Math. **197** (2006), no. 1, 53-127.
- [6] E. Calabi and B. Eckmann: A class of compact, complex manifolds which are not algebraic. Ann. of Math. (2) **58**, (1953), 494-500.
- [7] H. Cartan: Quotient d'un espace analytique par un groupe d'automorphismes. A symposium in honor of S. Lefschetz, Algebraic geometry and topology. pp. 90-102. Princeton University Press, Princeton, N. J. 1957.
- [8] A. Haefliger and D. Sundararaman: Complexifications of transversely holomorphic foliations. Math. Ann. **272** (1985), no. 1, 23-27.
- [9] T. Höfer: Remarks on torus principal bundles. J. Math. Kyoto Univ. **33** (1993), no. 1, 227-259.
- [10] G. Hochschild: The structure of Lie groups. Holden-Day, Inc., San Francisco-London-Amsterdam 1965.

- [11] H. Hopf: Zur Topologie der komplexen Mannigfaltigkeiten. Courant Anniversary Volume, New York, (1948), p. 168.
- [12] H. Samelson: A class of complex-analytic manifolds. *Portugaliae Math.* **12** (1953), 129-132.
- [13] H. Samelson: Topology of Lie groups. *Bull. Amer. Math. Soc.* **58**, (1952), 2-37.
- [14] E. B. Vinberg, V. V. Gorbatshevich and A. L. Onishchik: Lie groups and Lie algebras, III. Structure of Lie groups and Lie algebras. *Encyclopaedia of Mathematical Sciences*, 41. Springer-Verlag, Berlin, 1994.
- [15] J. J. Loeb and M. Nicolau: Holomorphic flows and complex structures on products of odd-dimensional spheres. *Math. Ann.* **306** (1996), no. 4, 781-817.
- [16] J. J. Loeb, M. Manjarin, and M. Nicolau: Complex and CR-structures on compact Lie groups associated to abelian actions. *Ann. Global Anal. Geom.* **32** (2007), no. 4, 361-378.
- [17] S. López de Medrano and A. Verjovsky: A new family of complex, compact, non-symplectic manifolds. *Bol. Soc. Brasil. Mat. (N.S.)* **28** (1997), no. 2, 253-269.
- [18] L. Meersseman: A new geometric construction of compact complex manifolds in any dimension. *Math. Ann.* **317** (2000), no. 1, 79-115.
- [19] M. Manjarin: Normal almost contact structures and non-Kähler compact complex manifolds. *Indiana Univ. Math. J.* **57** (2008), no. 1, 481-507.
- [20] L. Meersseman and A. Verjovsky: Holomorphic principal bundles over projective toric varieties. *J. Reine Angew. Math.* **572** (2004), 57-96.
- [21] A. L. Onishchik and É. B. Vinberg: Lie groups and algebraic groups. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. xx+328 pp.
- [22] T. Panov, and Y. Ustinovsky: Complex-analytic structures on moment-angle manifolds. *Mosc. Math. J.* **12** (2012), no. 1, 149-172, 216.
- [23] H. Pittie: The Dolbeault-cohomology ring of a compact, even-dimensional Lie group. *Proc. Indian Acad. Sci.* **98** (1988), pp. 117-152.
- [24] P. Sankaran and A. S. Thakur: Complex structures on product of circle bundles over complex manifolds. *Ann. Inst. Fourier (Grenoble)* **63** (2013), no. 4, 1331-1366.
- [25] T. A. Springer: Linear algebraic groups. Second edition. Progress in Mathematics, 9. Birkhuser Boston, Inc., Boston, MA, 1998.
- [26] D. N. Lee: The structure of complex Lie groups. Chapman & Hall/CRC Research Notes in Mathematics, 429. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [27] H.-C. Wang: Closed manifolds with homogeneous complex structure. *Amer. J. Math.* **76**, (1954). 1-32.
- [28] S. Cupit-Foutou and D. Zaffran: Non-Kähler manifolds and GIT-quotients. *Math. Z.* **257** (2007), no. 4, 783-797.
- [29] V. Vuletescu: LCK metrics on elliptic principal bundles, arXiv:1001.0936 [math.DG].
- [30] Z. Z. Wang and D. Zaffran: A remark on the hard Lefschetz theorem for Kähler orbifolds. *Proc. Amer. Math. Soc.* **137** (2009), no. 8, 2497-2501.

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